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# On Approximate Decentralization of Pareto Optimal Allocations in Locally Convex Spaces*

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We present approximate versions of the second fundamental theorem of welfare economics in the setting of dual pairs of locally convex spaces. We do not require the commodity space to have a positive cone with a nonempty interior. We also present a sufficient condition under which our approximate results hold exactly.


## 1. INTRODUCTION

In a pioneering paper, Debreu [4] extended the second fundamental theorem of welfare economics to economies with a topological vector space of commodities. Specifically, he showed that corresponding to a given Pareto optimal allocation of such an economy, there exists a price system at which the given allocation can be sustained as a valuation equilibrium. Debreu formalized the notion of a price system as a non-zero element of the topological dual of the commodity space and assumed, in particular, that the aggregate production set of the economy has a nonempty interior. Under this assumption and given convexity, Debreu could prove his result as a consequence of the supporting hyperplane theorem.

Recent work in mathematical economics (see Mas-Colell [11] and the references therein) has questioned Debreu's justification of the interiority assumption. This work has emphasized the importance of several spaces,

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none of which has a positive cone with a nonempty interior. Thus, the economic assumptions of "free disposal" or "monotonic preferences" do not lead, as in the case of [4] or in the subsequent work of Bewley [2], to a set which has a nonempty interior and which needs to be supported. Indeed, it now appears that from the point of view of economic theory, $R^n$ and $L_\infty$ may be the only interesting spaces for which this is true. Furthermore, we now have simple examples of economies for which Debreu's theorem does not hold; see Jones [8] and also [11].

Thus, there are two natural directions in which one can proceed. The first is to look for additional conditions on the underlying parameters of the economy under which Debreu's theorem can be proved. This is the approach of Mas-Colell [11, 12] and of subsequent work that utilizes his concept of "proper preferences." Alternatively, one can look for satisfactory approximate versions of Debreu's result as is done in [9] and in another context by Aliprantis, Brown, and Burkinshaw [1]. This is the direction that we pursue in this paper, although we also present a sufficient condition on an individual production set under which we obtain an exact result.

Our work uses in an essential way recent results in functional analysis. For our approximate results, the basic idea is to find an "approximate" support to the closure of a convex set with an empty interior. This then leads to the question of finding sufficient conditions that are attractive in terms of economic theory and under which the sum of closed convex sets is closed. We present two such conditions. The first is due to the collective efforts of Choquet [3], Dieudonné [6], and Ky Fan [7] and it allows us to set our results in real, Hausdorff locally convex spaces. The second condition leads us to restrict the commodity space to those spaces which are strictly hypercomplete [15, Sects. 12-2 and 12-3]. Our exact result, perhaps not surprisingly for functional analysts, draws on the Bishop–Phelps theorem, specifically a result [13] whose importance for economic theory was first seen by Majumdar [10]. More generally, however, our results bring out the relevance of functional analytic methods for a rather basic problem in mathematical economics.

The paper is organized as follows. Section 2 presents the model and results and Section 3 the proofs.

## 2. The Model and Results

For the basic terminology and notation we follow Wilansky [15]. However, for ease of exposition we recall some basic concepts for the reader.

Let $[E, F, v]$ be a *dual pair*, i.e., $E$, $F$ are vector spaces over the reals and $v$ is a bilinear functional on $E \times F$ and written $v(x, y) = [x, y]$ and such that

(a) $[x, y] = 0$ for all $y$ in $F$ implies $x = 0$ and

(b) $[x, y] = 0$ for all $x$ in $E$ implies $y = 0$.

In what follows, $E$ will be given the interpretation of a commodity space, $F$ that of a price space and $v$ a valuation functional. Thus $[x, y]$ denotes the value of a commodity bundle $x$ in $E$ at the price $y$ in $F$.

As is customary, we shall abbreviate a dual pair to be $[E, F]$ and assume that some bilinear functional is specified. In the special case that $F \subset E^*$, where $E^*$ is the algebraic dual of $E$, the bilinear functional is given by $\langle x, y \rangle = y(x)$ for any $x$ in $E$ and for any $y$ in $F$.

For any dual pair $[E, F]$, a Hausdorff locally convex topology $T$ on $E$ (resp. $F$) is said to be *compatible* with the dual pair if the topological dual of $E$, $E'$ (resp. $F'$), is $F$ (resp. $E$).

We shall respectively denote the weak, strong, and Mackey topologies, say, on $E$ as $\sigma(E, F)$, $\beta(E, F)$, and $\tau(E, F)$. We shall refer to the topology $\sigma(E', E)$ as the weak* topology and to $\beta(E', E)$ as the strong* topology. Thus, for example, a weak* closed subset of $E'$ should be taken to mean a $\sigma(E', E)$-closed subset of $E'$.

When we refer to a set satisfying a particular topological property without specifying the topology, it is to be understood that the said property holds in any compatible topology. In our view, it is one of the advantages of our approach that we can state the assumptions underlying our results in any compatible topology.

In the sequel we shall assume that $E$ is a partially ordered vector space and $F$ is endowed with an ordering induced by that on $E$. Given the recent work of Aliprantis *et al.* [1], it may be worth mentioning that neither $E$ nor $F$ is assumed to be a lattice.

We now turn to our basic economic concepts.

DEFINITION 1. An *economy* $\mathscr{E}$ is given by $\{[E,F], \geqslant, (X(t), \succsim_t, e(t))_{t \in T}, (Y(j))_{j \in P}\}$ where

(i) $[E, F]$ is a dual pair with $(E, \geqslant)$ a partially ordered vector space,

(ii) $T$ is a finite set of consumers such that for each consumer $t$, $X(t) \subset E$, $\succsim_t$ is a binary relation on $X(t) \times X(t)$ and $e(t) \in X(t)$,

(iii) $P$ is a finite set of producers such that for each producer $j$, $Y(j) \subset E$.

For any set $S$ of positive integers, let $|S|$ denote the cardinality of $S$.

DEFINITION 2. An *allocation* $(x, y) = \{(x(t))_{t \in T}, (y(j))_{j \in P}\}$ of $\mathscr{E}$ is a $(|T| + |P|)$-tuple such that

(i) $x(t) \in X(t)$ for all $t$ in $T$,

(ii) $y(j) \in Y(j)$ for all $j \in P$,

(iii) $\sum_{t \in T} x(t) = \sum_{j \in P} y(j) + \sum_{t \in T} e(t)$.

DEFINITION 3. An allocation $(x, y)$ of $\mathscr{E}$ is said to be *Pareto Optimal* if there does not exist any other allocation $(x', y')$ such that for all $t$ in $T$, $x'(t) \gtrsim_t x(t)$ and for at least one $t$, $x'(t) >_t x(t)$, where $a >_t b$ means $a \gtrsim_t b$ and not $b \gtrsim_t a$.

Without any confusion, we denote the induced ordering on $F$ also by $\geqslant$. For any $p, q$ in $F$, let $p > q$ denote $(p - q) \geqslant 0$ and $(p - q) \neq 0$. Let $N(E)$ (resp. $N(F)$) denote the set of neighborhoods of zero in $E$ (resp. $F$).

DEFINITION 4. For any real number $\delta > 0$ and any $z \in E_{-}/\{0\}$, a $(\delta, z)$-*approximate valuation equilibrium* of $\mathscr{E}$ consists of a pair $(p, (x, y))$ such that

(i) $p \in F$, $p > 0$ and $[z, p] = -1$.

(ii) $(x, y)$ is an allocation of $\mathscr{E}$ such that

  (a) $z \gtrsim_t x(t)$ implies $[z, p] \geqslant [x(t), p] - \delta$ for all $t$ in $T$,

  (b) $z \in Y(j)$ implies $[z, p] \leqslant [y(j), p] + \delta$ for all $j$ in $P$.

Thus, an approximate valuation equilibrium consists of a price system and an allocation such that

(i) each consumer is approximately minimizing expenditures and

(ii) each produces is approximately maximizing his profits.

In each case, the degree of approximation is measured by $\delta$ and $z$, the latter controlling for the fact the price system is not normalized to render this approximation trivial. If $\sum_{t \in T} e(t) \in E_{+}/\{0\}$, by letting $z = -\sum_{t \in T} e(t)$, we can consider approximate valuation equilibria with the normalization $[\sum_{t \in T} e(t), p] = 1$.

Our final concept relates to an economy modelled on the dual pair $(L', L'')$ where $L$ is a Banach space.

DEFINITION 5. A *valuation equilibrium* of $\mathscr{E}$ is a pair $(p, (x, y))$ such that

(i) $p \in L''$, $p > 0$, $\|p\| = 1$.

(ii) $(x, y)$ is an allocation of $\mathscr{E}$ such that

  (a) $z \gtrsim_t x(t)$ implies $\langle z, p \rangle \geqslant \langle x(t), p \rangle$ for all $t$ in $T$.

  (b) $z \in Y(j)$ implies $\langle z, p \rangle \leqslant \langle y(j), p \rangle$ for all $j$ in $P$.

It should be noted that the only difference between our definition of a valuation equilibrium and Debreu's definition in [4] is that we require expenditure minimization rather than preference maximization. Conditions under which the former implies the latter are well known. It should also be noted that the price system in Definition 5 is required to be in the topological dual and that it may be desirable to find conditions which ensure that it lies in the predual. To anticipate somewhat, in our approximate result (Theorem 3) we do indeed find $p$ in $L$, although our exact result (Theorem 4) requires $p$ to be in $L''$.

For our results we shall need the following assumptions on the economy $\mathscr{E} = \{[E, F], \geqslant, (X(t), \gtrsim_t, e(t))_{t\in T}, (Y(j))_{j\in P}\}$. For any closed, convex subset $K$ of $E$, $A(K)$ denotes the asymptotic cone of $K$ and, following [3], is given by $\bigcap_{\lambda>0} \lambda(K-x)$, where $x$ is any particular point in $K$. Let $E_+ = \{x \in E: x \geqslant 0\}$ and $E_- = -E_+$.

(A1) (a) For all $t$ in $T$, $X(t)$ is a closed convex subset of $E$.

(b) There exists convex $K \subset E$ such that for all $t$ in $T$, $X(t) \subset K$ and $K$ has a lower bound for $\geqslant$.

(A2) For all $t$ in $T$, for any $y$ in $X(t)$, the set $R_t(y) = \{x \in X(t): x \gtrsim_t y\}$ is closed, convex, and contains $y$.

(A3) (a) For all $j$ in $P$, $Y(j)$ is a closed, convex subset of $E$.

(b) For all $j$ in $P$, $0 \in Y(j)$ and $(E_-) \subset Y$, $\bar{Y} \cap (-\bar{Y}) = \{0\}$, where $Y = \sum_{j\in P} Y(j)$ and $\bar{Y}$ denotes the closure of $Y$.

(A4) There exists a $\tau$ in $T$ such that for any $x$, $y$, and $z$ in $X(\tau)$, $x > y$, $y >_\tau z \Rightarrow x >_\tau z$.

It is worth emphasizing that we do not assume that $\gtrsim_t$ is either complete or transitive.

For our first result, we shall need the following conditions in which for any $A \subset E$, $A^0 = \{p \in F: [x, p] \leqslant 1 \text{ for all } x \in A\}$.

Choquet Condition. The set $K$ in (A1)(b) and $Y$ are $\sigma(E, F)$-complete.

For our next condition, we shall adopt the convention throughout this paper that locally compact will refer to the $\sigma(E, F)$-topology.

Dieudonné Condition. The set $K$ in (A1)(b) and $Y$ are locally compact.

We can now state

Theorem 1. *If $\mathscr{E}$ satisfies (A1)–(A4) and if either the Choquet or the Dieudonné Condition is fulfilled, then for any $\delta > 0$, $z \in E_-/\{0\}$ and for any*

*Pareto optimal allocation* $(x^*, y^*)$ *of* $\mathscr{E}$, *there exists* $p^* \in F$, *such that* $(p^*, (x^*, y^*))$ *is a* $(\delta, z)$*-approximate valuation equilibrium of* $\mathscr{E}$.

COROLLARY 1. *Theorem* 1 *is valid if either the Choquet or the Dieudonné Condition is replaced by the requirement that the* $\tau(F, E)$*-interiors of* $K^0$ *and* $Y^0$ *are nonempty.*

The corollary makes clear the fact that the hypotheses relating to $K$ are no restriction for an economy modelled on the dual pair $[L', L]$, $L$ a Banach space such that $L_+$ has a nonempty norm interior. The hypotheses relating to $Y$ are, however, a restriction even in these circumstances and our next two results attempt to cope with this by restricting the class of locally convex spaces. They also require the following conditions on $X(t)$, $Y(j)$ and on the mutual position of the sum of these sets.

DIEUDONNÉ CONDITION II. (i) There exists $\alpha$ in $T$ such that $\sum_{t \neq \alpha} X(t)$ is locally compact.

(ii) There exists $f \in P$ such that $\sum_{j \neq f} Y(j)$ is locally compact.

CONDITION A. For any $\beta(E, F)$-bounded set $B$, $(\sum_{t \in T} X(t)) \cap (\sum_{j \in P} Y(j) + B)$ is $\beta(E, F)$-bounded.

We can now state

THEOREM 2. *Let* $\mathscr{E}$ *satisfy* (A1)–(A4), *Condition* A *and the Dieudonné Condition* II. *If* $\beta(E, F)$*-sequentially closed convex sets are* $\beta(E, F)$*-closed and* $[F, E]$ *is a quasibarrelled dual pair, then the conclusion of Theorem* 1 *holds.*

Our final result of this genre substitutes the following condition for the Dieudonné Condition II and, in so doing, allows us to drop (A1)(b) and (A3)(b).

KY FAN CONDITION. (i) For any Pareto optimal allocation $(x, y)$, $(R_t(x(t)))^0 \cap (\tau(F, E)\text{-interior } (\sum_{i=1}^{t-1} R_i(x(i)))^0) \neq \emptyset$, $t = 2, ..., |T|$.

(ii) $(Y(i))^0 \cap (\tau(F, E)\text{-interior } (\sum_{j=1}^{i-1} Y(j))^0) \neq \emptyset$, $i = 2, ..., |P|$.

We can now state

COROLLARY 2. *Theorem* 2 *is valid without*(A1)(b) *and* (A3)(b) *if the Ky Fan Condition is substituted for the Dieudonné Condition* II.

For our next result we model our economy $\mathscr{E}$ on the dual pair $[L', L]$, where $L$ is a real, locally convex Hausdorff space and $L'_+$ is a *normal* cone for $\beta(L', L)$, (see [14, p. 215] for a definition). Moreover, we shall need to substitute (A3′) for (A3), where

(A3′) $Y$ is a closed, convex subset of $L'$.

We shall also need

CONDITION B. For any $U \in N(L)$, $(\sum_{t\in T} X(t)) \cap ((\sum_{t\in T} e(t) + \sum_{j\in P} Y(j)) + U^0)$ is strong* bounded.

We can now state

THEOREM 3. *Let $\mathscr{E}$ satisfy* (A1), (A2), (A3′), *and* (A4). *If, in addition, $\mathscr{E}$ satisfies Condition* B *and L is strictly hypercomplete and quasibarrelled, then the conclusion of Theorem* 1 *holds.*

It is well known that a strictly hypercomplete quasibarrelled space is barrelled; see, for example, [15, Theorem 12-4-3 and Remark 10-4-13]. We do not know if Theorem 3 is true for barrelled spaces.

Our final result presents a sufficient condition under which Pareto optimal allocations can be exactly decentralized. This sufficient condition formalizes the requirement of uniformly bounded marginal rates of substitution of any one production set. We consider an economy $\mathscr{E}$ modelled on the dual pair $(L', L'')$.

CONDITION M. There exists $f$ in $P$ and there exists $\bar{p} \in L''$, $\bar{p} > 0$ such that any supporting hyperplane with unit norm, $p \in L''$, $\|p\| = 1$, to the set $Y(f)$ satisfies $p \geqslant \bar{p}$.

Condition $M$ is an assumption on the parameters of an individual agent and does not assert that a supporting hyperplane necessarily exists at a particular boundary point of $Y(f)$.

THEOREM 4. *Let $\mathscr{E}$ be modelled on $(L', L'')$ with L a Banach space and let $\mathscr{E}$ satisfy* (A1), (A2), (A3′), (A4), *and Conditions* B *and* M. *Then corresponding to any Pareto optimal allocation $(x^*, y^*)$ of $\mathscr{E}$, there exists $p^* \in L$ such that $(p^*, (x^*, y^*))$ is a valuation equilibrium of $\mathscr{E}$.*

*Remark* 2. Note that Theorem 4 has nothing to say about an exchange economy, i.e., one, where $Y(j) = \{0\}$ for all $j$.

*Remark* 3. We have presented our exact result in the context of the setting of Theorem 4. It should be clear from an inspection of the proofs that we could equally well have presented it under the setting of our other approximate results, i.e., with the Choquet or the Dieudonné conditions instead of Condition B.

## 3. PROOFS

We begin with the following elementary result.

LEMMA. *Let $Z_1$ and $Z_2$ be two subsets of E. Then*

(a) *for any real number* $\lambda \neq 0$, $\lambda(Z_1 \cap Z_2) = \lambda Z_1 \cap \lambda Z_2$;

(b) *for any z in E,* $(Z_1 \cap Z_2) - (z) = (Z_1 - z) \cap (Z_2 - z)$;

(c) *if $Z_1$ and $Z_2$ are closed convex and such that* $Z_1 \cap Z_2 \neq 0$, $A(Z_1 \cap Z_2) = A(Z_1) \cap A(Z_2)$.

*Proof of the Lemma.* We begin with (a). For any $z$ in $\lambda(Z_1 \cap Z_2)$, there exists $k$ in $Z_1 \cap Z_2$ such that $z = \lambda k$ which implies $z$ is in $(\lambda Z_1 \cap \lambda Z_2)$. On the other hand, if $z$ is in $\lambda Z_1 \cap \lambda Z_2$, there exist $z_i \in Z_i$ such that $z = \lambda z_1 = \lambda z_2$. Since $\lambda \neq 0$, $z_1 = z_2$ and hence $z \in \lambda(Z_1 \cap Z_2)$.

As regards (b), $k \in (Z_1 \cap Z_2) - (z)$ if and only if $(k + z) \in Z_i$ $(i = 1, 2)$, if and only if $k \in Z_i - z$ $(i = 1, 2)$, that is $k \in (Z_1 - z) \cap (Z_2 - z)$.

Let $k \in Z_1 \cap Z_2$. Then by (a) and (b), $z \in [\bigcap_{\lambda > 0} \lambda(Z_1 - k)] \cap [\bigcap_{\lambda > 0} \lambda(Z_2 - k)]$ if and only if $z \in [\bigcap_{\lambda > 0} \lambda((Z_1 \cap Z_2) - k)]$. ∎

*Proof of Theorem* 1. Let $(x^*, y^*)$ be a Pareto optimal allocation of $\mathscr{E}$ and let

$$W = \sum_{t \in T} R_t(x^*(t)) - \sum_{j \in P} Y(j) - \sum_{t \in T} x^*(t) + \sum_{j \in P} y^*(j),$$

where $R_t(x^*(t))$ is as defined in (A2).

The essential part of the proof is to show that $W$ is a closed set in $E$. We begin with the case when the Choquet Condition holds. Since $K$ is bounded from below for $\geqslant$, there exists $b \in E$ such that $K - b$ is a subset of $E_+$. By the Choquet Condition, $K - b$ is a $\sigma(E, F)$-complete subset of $E$. Since $E$ is an ordered vector space, $E_+$ contains no straight line. By (A2), $R_t(x^*(t))$ is closed, convex for all $t$ in $T$. We can now appeal to Choquet's theorem [3] to assert that $\sum_{t \in T} R_t(x^*(t))$ is closed. Next we show that $\bar{Y}$ contains no straight line. Suppose it did, i.e., there exist $x$, $y$ in $E$, $x \neq y$ such that $\lambda x + (1 - \lambda) y$ is in $\bar{Y}$ for all real numbers $\lambda$. This implies $(1 - \lambda)(y - x) \in A(\bar{Y})$ for all $\lambda$. On choosing $\lambda = 0$ and $\lambda = 2$, we can conclude that $(y - x)$ is in $A(\bar{Y})$ and in $-A(\bar{Y}) = A(-\bar{Y})$. On using (c) of the lemma, we contradict the fact that $\bar{Y} \cap (-\bar{Y}) = \{0\}$. Hence by a second appeal to Choquet's theorem, we can assert that $\sum_{j \in P} Y(j)$ is closed. For the final step, observe that (A3)(b) implies that $E_+ \subset -Y$. Since $(K - b) \subset E_+$, $K \subset \mathrm{b} - Y$. Since $\sum_{t \in T} R_t(x^*(t))$ is contained in $K$, we can assert that the set $(b - Y)$ contains $\sum_{t \in T} R_t(x^*(t))$. Since $Y$ is $\sigma(E, F)$-complete by the Choquet condition, so is $b - Y$. We can now make a third appeal to Choquet's theorem to assert that $\sum_{t \in T} R_t(x^*(t)) - Y + b$ is closed. This implies that $W$ is closed and completes the proof of our claim.

Next, we turn to the case when the Dieudonné Condition holds. Since $R_t(x^*(t))$ are closed subsets of a locally compact set, they are locally compact. Since $R_t(x^*(t))$ are subsets of $K$ and $K$ is bounded from below for $\geqslant$, there exists $b \in E$ such that $R_t(x^*(t)) - b \subset E_+$ for all $t$ in $T$. Since $R_t(x^*(t)) - b$ is closed and convex, we can take its asymptotic cone. Since $E_+ \cap E_- = \{0\}$, $A(R_t(x^*(t)) \cap A(-R_s(x^*(s)) = \{0\}$ for any $t, s$ in $T$. We can now appeal to Dieudonné's theorem [6] to assert that $R_t(x^*(t)) + R_s(x^*(s))$ is closed for any $t, s$ in $T$. The same argument can be repeated for a given $R_u(x^*(u))$ and $R_t(x^*(t)) + R_s(x^*(s))$. Proceeding in this way, we can show that $\sum_{t \in T} R_t(x^*(t))$ is closed.

Since $Y(j) \subset Y$, $Y(j)$ is closed, and $Y$ is locally compact, $Y(j)$ is locally compact for $j \in P$. Furthermore $Y(i) \cap (-Y(j)) = \{0\}$ for all $i \neq j$ in $P$ as a consequence of $\bar{Y} \cap (-\bar{Y}) = \{0\}$. Hence, by a second appeal to Dieudonné's theorem, we can assert that $Y(i) + Y(j)$ is closed. By repeating this argument for another $Y(k)$, we can show that $\sum_{j \in P} Y(j)$ is closed.

Next, we assert that $Y \cap E_+ = \{0\}$. If not, there exists $y$ in $Y$, $y \in E_+$ and such that $y \neq 0$. By (A3)(b), $-y \in Y$. This implies $y \in (-Y)$ and we contradict the fact that $\bar{Y} \cap (-\bar{Y}) = \{0\}$. Since $K - b \subset E_+$, so is its closure $\bar{K} - b$. This implies that $Y \cap (\bar{K} - b) = \{0\}$ which implies that $A(Y) \cap A(\bar{K}) = \{0\}$. Since $A(\sum_{t \in T} R_t(x^*(t))) \subset A(\bar{K})$, and $Y$ is locally compact, a final appeal to Dieudonné's theorem completes the proof of our claim that $W$ is closed.

Next, we claim that $W \cap E_- = \{0\}$. This is a simple consequence of (A4) and the fact that $(x^*, y^*)$ is a Pareto optimal allocation of $\mathscr{E}$.

By (A2) and (A3), $W$ is a nonempty, convex set. Given any $z \in E_- / \{0\}$, we can apply the second separation theorem [13, II.9.2] to assert the existence of $p \in F$, $p \neq 0$ such that

$$[w, p] > [\delta z, p] \qquad \text{for all} \quad w \in W. \tag{1}$$

Since $0 \in W$, $[\delta z, p] < 0$. Now let $p^* = -\delta p / [\delta z, p]$ so that $p^* z = -1$. Then we can rewrite (1) as

$$[w, p^*] > -\delta \qquad \text{for all} \quad w \in W. \tag{2}$$

Now for any $t$ in $T$, and any $x$ in $R_t(x^*(t))$, let $v = (x - x^*(t))$. Certainly $v \in W$, and hence

$$[x, p^*] > [x^*(t), p^*] - \delta. \tag{3}$$

Similarly, for any $j \in P$, and any $y \in Y(j)$, $u = (-y + y^*(j))$ is an element of $W$. This yields

$$[y, p^*] < [y^*(j), p^*] + \delta. \tag{4}$$

Finally, we show that $p^* > 0$. Suppose there exists $v$ in $E_+$ such that $\langle v, p^* \rangle < 0$. By (A4), $(x^*(\tau) + kv) \in R_\tau(x^*(\tau))$ for any positive integer $k$. This implies $k[v, p^*] > -\delta$ for any $k$, an obvious contradiction. ▮

*Proof of Corollary* 1. Since Ky Fan's Theorem 1 [7] shows that nonempty $\tau(F, E)$-interiority of $K^0$ implies that $K$ is $\sigma(E, F)$-complete and locally compact, assuming the former property for $K$ and $Y$ requires no changes in the proof. ▮

*Proof of Theorem* 2. Let $W$ be defined exactly as in the proof of Theorem 1. Our first claim is that $W$ is $\beta(E, F)$-closed. Since $W$ is convex, we need only show that $W$ is $\beta(E, F)$-sequentially closed. Towards this end, choose a sequence $\{w^n\}$ from $W$ such that the $\beta(E, F)$-limit of $w^n$ is $w$. We have to show $w \in W$. Since $w^n \in W$, there exist $x^n(t) \in R_t(x^*(t))$, $y^n(j) \in Y(j)$ such that

$$\sum_{t \in T} (x^n(t) - x^*(t)) - \sum_{j \in P} (y^n(j) - y^*(j)) = w^n. \tag{5}$$

Since $\{w^n\}$ is a $\beta(E, F)$-bounded set, there exists a $\beta(E, F)$-bounded set $B$ such that

$$\left(w^n + \sum_{t \in T} x^*(t) - \sum_{j \in P} y^*(j)\right) \in B \qquad \text{for all } n. \tag{6}$$

This implies that

$$\left(\sum_{t \in T} x^n(t)\right) \in \left(\sum_{t \in T} X(t)\right) \cap (Y + B). \tag{7}$$

We can now appeal to condition A to assert that $\{\sum_{t \in T} x^n(t)\}$ is a $\beta(E, F)$-bounded set. Since $F$ is quasibarrelled in any compatible topology, we can apply [15, Theorem 10–10–11] to assert that $\{\sum_{t \in T} x^n(t)\}$ is $\tau(F, E)$ equicontinuous and hence, by Alaoglu–Bourbaki, $\sigma(E, F)$-relatively compact. This implies that there exists a subnet $\{\sum_{t \in T} x^{\nu}(t)\}$ of $\{\sum_{t \in T} x^n(t)\}$ which converges in $\sigma(E, F)$ to a limit, say $\bar{x}$. It is clear from an inspection of the relevant arguments in the proof of Theorem 1, that our appeal to Diedonné's theorem requires local compactness of all but one of $R_t(x^*(t))$. Hence, $\sum_{t \in T} R_t(x^*(t))$ is closed by Dieudonné Condition II(i). Hence $\bar{x} \in \sum_{t \in T} R_t(x^*(t))$.

We can now rewrite (5) as

$$\sum_{j \in P} y^{\nu}(j) = \sum_{t \in T} (x^{\nu}(t) - x^*(t)) + \sum_{j \in P} y^*(j) + w^{\nu}. \tag{8}$$

On taking $\sigma(E, F)$-limits of both sides and on observing that $\sigma(E, F)$-limit of $\{w^{\nu}\}$ is $w$, we can show that $\sum_{j \in P} y^{\nu}(j)$ also has a $\sigma(E, F)$-limit, say $\bar{y}$. However, as in the proof of Theorem 1, a second application of Dieudonné's theorem along with Dieudonné Condition II(ii) and (A3) yields the fact that $\sum_{j \in P} Y(j)$ is closed. Hence there exist $\bar{y}(j) \in Y(j)$ such that $\sum_{j \in P} \bar{y}(j) = \bar{y}$. This implies

$$w = \bar{x} - \sum_{j \in P} \bar{y}(j) - \sum_{t \in T} x^*(t) + \sum_{j \in P} y^*(j), \tag{9}$$

and completes the proof of the assertion that $w \in W$.

We can now follow the remaining steps in the proof of Theorem 1 to complete the proof of the theorem. ∎

*Proof of Corollary* 2. Given Ky Fan Condition (i), we can make successive appeals to Theorem 1 in [7] to assert that $\sum_{t \in T} R_t(x^*(t))$ is closed. Also given Ky Fan Condition (ii), successive appeals to Theorem 1 in [7] allows us to conlude that $Y$ is closed. We can now repeat the argument in the proof of Theorem 2 and conclude that $\bar{x} \in \sum_{t \in T} R_t(x^*(t))$ and $\bar{y} \in \sum_{j \in P} Y(j)$ without any appeal to Dieudonné Condition II. The remaining steps in the proof of Theorem 2 which call for the repetition of the corresponding steps in the proof of Theorem 1, remain unchanged. ∎

*Proof of Theorem* 3. Let $W$ be defined exactly as in the proof of Theorem 1. Our first claim is that $W$ is weak* closed. Since $L$ is strictly hypercomplete, all we need to show is that $W \cap U^0$ is weak* closed for any $U \in N(L)$, [15, 12-3-7 and 12-2-3]. Towards this end, pick any net $\{w^\nu\}$ from $W \cap U^0$ and such that the weak* limit of $\{w^\nu\}$ is $w$. We have to show that $w \in W \cap U^0$. Since $w^\nu \in W$, there exist $x^\nu(t) \in R_t(x^*(t))$, $y^\nu(j) \in Y(j)$ such that

$$\sum_{t \in T} x^\nu(t) = w^\nu + \sum_{j \in P} y^\nu(j) + \sum_{t \in T} x^*(t) - \sum_{j \in P} y^*(j). \tag{10}$$

Since $w^\nu \in U^0$, and since $(x^*, y^*)$ is an allocation, we can appeal to Condition B to assert that $\{\sum_{t \in T} x^\nu(t)\}$ is strong* bounded. We can apply [15, Theorem 10-1-11] to assert that $\{\sum_{t \in T} x^\nu(t)\}$ is equicontinuous. By Alaoglu–Bourbaki, this implies that there exists a weak* convergent subnet $\{\sum_{t \in T} x^\rho(t)\}$. Let the weak* limit of this subnet be $\bar{x}$.

Since $\{\sum_{t \in T} x^\nu(t)\}$ is a strong* bounded set, so is $\{\sum_{t \in T} x^\rho(t)\}$. Furthermore, since $x^\rho(t) \in X(t)$, there exists $b \in E$ such that $(x^\rho(t) - b) \geqslant 0$ for all $t$ in $T$ and all $\rho$. Since $L'_+$ is a normal cone, we can appeal to [14, Theorem 3.1, p. 215] to assert that $\{x^\rho(t)\}$ is strong* bounded. Since $L$ is quasibarrelled, by a second appeal to [15, Theorem 10-1-11], we can assert that $\{x^\rho(t)\}$ is equicontinuous. By Alaoglu–Bourbaki, this implies that there exists a weak* convergent subnet, also denoted by $\{x^\rho(t)\}$, which tends to a limit $x'(t)$. Since $\{x^\rho(t)\}$ is a net from $R_t(x^*(t))$ and the latter is a closed set, $x'(t) \in R_t(x^*(t))$. By taking as many subnets as necessary, we can conclude that $\sum_{t \in T} x'(t) \in \sum_{t \in T} R_t(x^*(t))$ and that $\bar{x} = \sum_{t \in T} x'(t)$.

Now we can rewrite (10) as

$$\sum_{j \in P} y^\rho(j) = -w^\rho + \sum_{t \in T} (x^\rho(t) - x^*(t)) + \sum_{j \in P} y^*(j). \tag{11}$$

Since the right-hand side tends to a weak* limit, the left-hand side tends to a weak* limit, say $\bar{y}$. Since $\sum_{j \in P} Y(j)$ is weak* closed by (A3′), it contains $\bar{y}$. This implies that

$$w = \bar{x} - \bar{y} - \sum_{t \in T} x^*(t) + \sum_{j \in P} y^*(j), \tag{12}$$

which in turn leads us to conclude that $w \in W \cap U^0$.

We can now follow the remaining steps in the proof of Theorem 1 to complete the proof. ∎

*Proof of Theorem* 4. Since $L$ is strictly hypercomplete [15, 12-3-3], we can proceed just as in the proof of Theorem 4 to assert that $W$ is weak* closed and that 0 is in the norm boundary of $W$. Since $L$ is a Banach space, so is $L'$. If the convex set $W$ can be supported at 0, the proof can be easily completed. If not, we can appeal to Phelps' theorem [13, Theorem 1] to assert, for each $\varepsilon > 0$, the existence of $w_\varepsilon \in W$, $\|w_\varepsilon\| < \varepsilon$, and $p_\varepsilon \in L/\{0\}$ such that

$$\langle w, p_\varepsilon \rangle \geqslant \langle w_\varepsilon, p_\varepsilon \rangle \qquad \text{for all} \quad w \in W. \tag{13}$$

Since $w_\varepsilon \in W$, there exist $x_\varepsilon(t) \in R_t(x^*(t))$ and $y_\varepsilon(j) \in Y(j)$ such that

$$\langle w, p_\varepsilon \rangle \geqslant \left\langle \sum_{t \in T} x_\varepsilon(t) - \sum_{j \in P} y_\varepsilon(j), p_\varepsilon \right\rangle$$
$$\text{for all} \qquad w \in \left( \sum_{t \in T} R_t(x^*(t)) - \sum_{j \in P} Y(j) \right). \tag{14}$$

Now consider the producer $f$ described in Condition M and deduce from (14) by appropriate choice of $w$ that

$$\langle z, p_\varepsilon \rangle \leqslant \langle y_\varepsilon(f), p_\varepsilon \rangle \qquad \text{for all} \quad z \in Y(f). \tag{15}$$

This implies that $p_\varepsilon$ supports $Y(f)$ at $y_\varepsilon(f)$. Without loss of generality we can assume $\|p_\varepsilon\| = 1$ and thus by Condition M, $p_\varepsilon \geqslant \bar{p} > 0$.

Since the above argument is true for all $\varepsilon > 0$ and since $p_\varepsilon$ has unit norm, $\{p_\varepsilon\}_{\varepsilon > 0}$ has a weak* convergent subnet with limit $p^*$. As $p_\varepsilon \geqslant \bar{p}$, $p^* \neq 0$.

Furthermore, as $\varepsilon$ tends to zero, $w_\varepsilon$ tends to zero in norm. Thus, we can rewrite (13) as

$$\langle w, p^* \rangle \geqslant 0 \qquad \text{for all} \quad w \in W. \tag{16}$$

We can now normalize $p^*$ to have unit norm and also appeal to (A4) to show that $p^* \in L''_+$. Finally, obvious computations complete the proof. ∎

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